And again problem related to quadratically connected sequences.
https://www.linkedin.com/feed/update/urn:li:activity:6577421871774482432
Let $a_n, b_n, n = 1,2,\ldots$ be two sequences defined by $a_1 = 1, b_1 = 0$ and for $n \geq 1$
$$\begin{cases} a_{n+1} = 7a_n + 12b_n + 6 \\ b_{n+1} = 4a_n + 7b_n + 3 \end{cases}.$$
Prove that a_n^2 is the difference of two consecutive cubes.
Solution by Arkady Alt, **San Jose**, **California**, **USA**.
From the system $\begin{cases} 1 = 7a_0 + 12b_0 + 6 \\ 0 = 4a_0 + 7b_0 + 3 \end{cases}$ we obtain $a_0 = 1$.
Also, since $12b_n = a_{n+1} - 7a_n - 6, n \in \mathbb{N}$ then $b_{n+1} = 4a_n + 7b_n + 3 \Leftrightarrow$
$12b_{n+1} = 48a_n + 7 \cdot 12b_n + 36 \Leftrightarrow a_{n+2} - 7a_{n+1} - 6 =$
$48a_n + 7(a_{n+1} - 7a_n - 6) + 36 \Leftrightarrow a_{n+2} - 14a_{n+1} + a_n = 0, n \in \mathbb{N}$.
Thus, sequence $(a_n)_{n \in \mathbb{N} \cup \{0\}}$ completely defined by recurrence
(**1**) $a_{n+1} - 14a_n + a_{n-1} = 0, n \in \mathbb{N}$ and initial conditions $a_0 = a_1 = 1$.
Noting that $a_2 = 14a_1 - a_0 = 13$ and $a_{n+2}a_n - a_{n+1}^2 =$
$(14a_{n+1} - a_n)a_n - a_{n+1}(14a_n - a_{n-1}) = a_{n+1}a_{n-1} - a_n^2, n \in \mathbb{N}$
we obtain that $a_{n+1}a_{n-1} - a_n^2 = a_2a_0 - a_1^2 = 13 - 1 = 12 \Leftrightarrow a_{n+1}a_{n-1} = a_n^2 + 12$
and, therefore, $(a_{n+1} + a_{n-1})^2 = 196a_n^2 \Leftrightarrow a_{n+1}^2 + a_{n-1}^2 + 2(a_n^2 + 12) = 96a_n^2 \Leftrightarrow$
(**2**) $a_{n+1}^2 - 194a_n^2 + a_{n-1}^2 = -24, n \in \mathbb{N}$.
We will prove that there is a sequence $(c_n)_{n \in \mathbb{N} \cup \{0\}}$ of integer numbers such that
$a_n^2 = (c_n + 1)^3 - c_n^3 \Leftrightarrow a_n^2 = 3c_n^2 + 3c_n + 1$ for any $n \in \mathbb{N} \cup \{0\}$ (obvious that $c_0 = -1$
and $c_1 = 0$ satisfies to this correlation)
First note that $a_n^2 \equiv 1 \pmod 3$ for any $n \in \mathbb{N} \cup \{0\}$.
For $n = 0, 1$ it obviously holds. For any $n \in \mathbb{N}$ assuming $a_{n-1}^2 \equiv 1 \pmod 3$ and
$a_n^2 \equiv 1 \pmod 3$
we obtain $a_{n+1}^2 = 194a_n^2 - a_{n-1}^2 \equiv 2a_n^2 - a_{n-1}^2 \pmod 3 \equiv (2 \cdot 1 - 1) \pmod 3 = 1 \pmod 3$.
Thus, by Math Induction $a_n^2 \equiv 1 \pmod 3$ for any $n \in \mathbb{N} \cup \{0\}$.
Since $a_n^2 = 3c_n^2 + 3c_n + 1 \Leftrightarrow \dfrac{a_n^2 - 1}{3} = c_n^2 + c_n \Leftrightarrow \dfrac{4(a_n^2 - 1)}{3} + 1 = \dfrac{4a_n^2 - 1}{3} = (2c_n + 1)^2$
then denoting $d_n := \dfrac{4a_n^2 - 1}{3}$, $n \in \mathbb{N} \cup \{0\}$ we obtain $d_0 = d_1 = 1, d_2 = \dfrac{4 \cdot 13^2 - 1}{3} = 15^2$
and by substitution $a_n^2 = \dfrac{3d_n + 1}{4}$ in the recurrence (**2**) we obtain
$\dfrac{3d_{n+1} + 1}{4} - 194 \cdot \dfrac{3d_n + 1}{4} + \dfrac{3d_{n-1} + 1}{4} = -24 \Leftrightarrow d_{n+1} - 194d_n + d_{n-1} = 32, n \in \mathbb{N}$.
We will prove that for sequence $(t_n)_{n \in \mathbb{N} \cup \{0\}}$ of integers defined by recurrence
(**3**) $t_{n+1} - 14t_n + t_{n-1} = 0, n \in \mathbb{N}$ and $t_0 = -1, t_1 = 1$ holds $d_n = t_n^2$, $\forall n \in \mathbb{N} \cup \{0\}$.
As above, noting that $t_2 = 14t_1 - t_0 = 15$ we obtain $t_{n+1}t_{n-1} - t_n^2 = t_2t_0 - t_1^2 =$
$15 \cdot (-1) - 1 = -16 \Leftrightarrow t_{n+1}t_{n-1} = t_n^2 - 16$ and, therefore,
$(t_{n+1} + t_{n-1})^2 = 196t_n^2 \Leftrightarrow t_{n+1}^2 + t_{n-1}^2 + 2(t_n^2 - 16) = 96t_n^2 \Leftrightarrow$
(**4**) $t_{n+1}^2 - 194t_n^2 + t_{n-1}^2 = 32, n \in \mathbb{N}$.
Having in account that $t_n > 0, n \in \mathbb{N}$ we have $\Leftrightarrow 2c_n + 1 = t_n, n \in \mathbb{N},$.

Note that $2c_n + 1 = t_n$ implies $(2c_n + 1)^2 = t_n^2$ for any $n \in \mathbb{N} \cup \{0\}$.
Then by substitution t_n in (**3**) we obtain
$(2c_{n+1} + 1) - 14(2c_n + 1) + (2c_{n-1} + 1) = 0 \Leftrightarrow c_{n+1} - 14c_n + c_{n-1} = 6.$
Thus, for sequence $(c_n)_{\mathbb{N}\cup\{0\}}$ of integers defined by recurrence
(**5**) $c_{n+1} - 14c_n + c_{n-1} = 6, n \in \mathbb{N}$ and $c_0 = -1, c_1 = 0$ for any $n \in \mathbb{N} \cup \{0\}$
holds $a_n^2 = (c_n + 1)^3 - c_n^3$.
Example how it works:
$a_1^2 = 1$ and $(c_1 + 1)^3 - c_1^3 = 1, a_2^2 = 13^2$ and $(c_2 + 1)^3 - c_2^3 = 8^3 - 7^3 = 13^2$,
$a_3 = 14 \bullet a_2 - a_1 = 14 \bullet 13 - 1 = 181, c_3 = 14 \bullet c_2 - c_1 = 14 \bullet 7 - 0 + 6 = 104 = 98$ and
$a_3^2 = 181^2,\ (c_3 + 1)^3 - c_3^3 = 105^3 - 104^3 = 181^2$.